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Transient heat conduction in the composite slab-analytical method

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Abstract
In this paper, a novel analytical approach to heat conduction in a composite slab subject to periodic temperature changes has been developed. Taking advantage of the periodic properties of boundary changes, the corresponding analytical solution is obtained and expressed explicitly. Unlike most of the traditional methods, it involves no residue evaluation and no iterative computation such as a numerical search for eigenvalues. The adopted method is simple and elegant. The physical parameters are clearly shown in the mathematical formula of the solution. Furthermore, comparison of the method with numerical calculations demonstrates the applicability and accuracy of the method.

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Notation

f	general function
g	general function
j	general index number
k	diffusivity
m	inverse Laplace transform of M
M	function defined in equation (3.9)
t	time
T	temperature
x	space coordinate

Greek letters

Δ	matrix determinant defined in equations (3.6) and (3.7)
ϕ	phase
φ	phase
λ	thermal conductivity
ϖ	period
ω	period

Superscripts

in	indoor
out	outdoor

1. Introduction

Transient heat conduction through a composite slab is of interest in various engineering applications. Mathematically it is completely analogous to mass diffusion. In construction applications, the problem of predicting thermal performances of buildings has attracted researchers for decades. The problem is complex because envelopes of buildings are often constructed with multi-layers and the heat equations governing the system have not been analytically solved. In classical works, analytical solutions are mostly obtained for a single material layer using finite integral transform [1, 2]. Techniques such as Laplace transform, Green's function and orthogonal expansion are also often adopted for obtaining analytical solutions. It is possible to use similar techniques for a composite slab (e.g. [3]). However, the associated eigenvalue problem becomes much more complicated [4].

One way to avoid calculating the eigenvalues is to focus on the solutions which only provide information about conductivity and time lag, i.e. the steady-state rate of the heat conduction and the time required for the attainment of steady state [5, 6]. With these two parameters, it is possible to estimate the total amount of heat loss and gain. Such an estimation is adequate for many practical applications. For heat conduction applications, de Monte [7] provided a detailed introduction in various solution methods for composite slabs. In mass diffusion applications, Fredman [4] reviewed various analytical approaches for composite materials. In addition, Fredman developed a semi-analytical solution for cylindrical geometry from the local solution on single material layer in combination with a numerical scheme for material boundary states.

It can be concluded that the analytical solutions of transient conduction on composite slabs are indeed very complicated, as pointed out by all the researchers cited above. Even for the simplest case of two-layer slab, solutions are too complicated for practical use [6]. Only recently, Sun *et al* [8] presented a solution of transient heat conduction on a one-dimensional three-layer composite slab subject to unchanged temperatures on the boundary. Their work was mainly based on the method developed recently by de Monte [7, 9] for a two-layer composite slab. A full series solution was found by employing a 'natural' orthogonal relationship between the eigenfunctions. The eigenvalue problem was solved with a numerical procedure, i.e. Newton iteration. Numerical and analytical calculations were compared. Sun *et al* [8] also conjectured partial solutions for an n-layer composite slab based on their previous analysis.

The adopted 'natural' eigenfunction expansion method in the previous work is straightforward. However, with more layers in the slab, the numerical work for the eigenvalue problem is more tedious and thus practically impossible to solve. Moreover, information is needed for many transient problems about a composite slab subject to inlet temperature varying with time periodically. In construction applications for example, average daily, monthly and yearly temperatures are usually given as input data. Boundary temperatures are often expressed as a sinusoidal excitation. The temperature variation of a building under such a boundary condition is important in analysing its thermal behaviour. It is desirable that the analysis is able to cope with these issues. Motivated by this point, we intend to derive an analytical solution in this paper for an n-layer composite slab subject to arbitrary periodic temperature changes.

Compared to the work reviewed above, firstly, the boundary condition is more relaxed and general. Second, there is no need to numerically search for the eigenvalues and no need to evaluate the residues. The adopted method is efficient, simple and elegant and the analytical solution is concise and easy to apply. The physical parameters are clearly shown in the mathematical formula. To assess the accuracy of the method, further comparison of the results with numerical calculations is also presented.

2. Mathematical problem

Consider an n-layer composite slab which has constant thermal conductivity, diffusivity and density for each layer. The thermal conductivity, diffusivity and thickness of each layer are presented as λ_j, k_j and l_j, $j = 1, \ldots, n$. Therefore the layers consist of regions $[0, l_1]$, $[l_1, l_2]$ and $[l_{n-1}, l_n]$. The boundaries are at $x = 0$ and $x = l_1 + \cdots + l_n$. The general heat conduction in the slab with convective boundary conditions can be described by the following equations for temperatures $T_j(t, x)$, $j = 1, \ldots, n$:

$$\frac{\partial T_1(t,x)}{\partial t} = k_1 \frac{\partial T_1^2(t,x)}{\partial x^2}, \qquad x \in [0, l_1], \tag{2.1a}$$

$$\frac{\partial T_j(t,x)}{\partial t} = k_j \frac{\partial T_j^2(t,x)}{\partial x^2}, \qquad x \in [l_1 + \cdots + l_{j-1}, l_1 + \cdots + l_j], \quad j = 2, \ldots, n \tag{2.1b}$$

and the boundary conditions are

$$-\lambda_1 \frac{\partial T_1}{\partial x}(t, 0) = -\alpha_{\text{out}}(T_1(t,0) - T_{\text{out}}(t)), \tag{2.2a}$$

$$T_j(t, l_1 + \cdots + l_j) = T_{j+1}(t, l_1 + \cdots + l_j), \qquad j = 1, \ldots, n-1, \tag{2.2b}$$

$$-\lambda_j \frac{\partial T_j}{\partial x}(t, l_1 + \cdots + l_j) = -\lambda_{j+1} \frac{\partial T_{j+1}}{\partial x}(t, l_1 + \cdots + l_j), \qquad j = 1, \ldots, n-1, \tag{2.2c}$$

$$-\lambda_n \frac{\partial T_n}{\partial x}(t, l_1 + \cdots + \lambda_n) = -\alpha_{\text{in}}(T_{\text{in}}(t) - T_n(t, l_1 + \cdots + \lambda_n)). \tag{2.2d}$$

The initial temperatures are $T_1(0, x) = 0$, $x \in [0, l_1]$ and $T_j(0, x) = 0$, $x \in [l_1 + \cdots + l_{j-1}, l_1 + \cdots + l_j]$, $j = 2, \ldots, n$. Here the convection heat transfer coefficients for outdoor and indoor boundaries are denoted as α_{out} and α_{in}. Outdoor and indoor temperatures are given as $T_{\text{out}}(t)$ and $T_{\text{in}}(t)$. Initially the indoor temperature is assumed to be constant. Since we are not interested in the initial temperatures of the slab, the initial temperatures do not need to be specified. We set initial data to zero for the sake of calculational convenience.

Without loss of generality, boundary temperatures are assumed to be

$$T_{\text{out}}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(\omega_k t + \varphi_k), \tag{2.3}$$

$$T_{\text{in}}(t) = 0. \tag{2.4}$$

Often, the outdoor temperature $T_{\text{out}}(t)$ is written as a simple sinusoidal wave, for example

$$T_{\text{out}}(t) = 5.6 - 10.7 \cos\left(\frac{2\pi}{365}(t - 20.0)\right) {}^{\circ}\mathrm{C} \tag{2.5}$$

is a good representation of daily outdoor temperatures in Helsinki area.

Through this paper, if there is no danger of confusion we shall only write the simple forms of all the notation. For example T_j, $T_j(l_1 + \cdots + l_n)$ instead of $T_j(t, x)$, $T_j(t, l_1 + \cdots + l_n)$ and T_{out} instead of $T_{\text{out}}(t)$.

3. Analytical solutions of the equations

3.1. Laplace transformation of the equations

To solve equation (2.1), we perform Laplace transformation. It yields

$$s\overline{T_1}(s, x) = k_1 \frac{\partial^2 \overline{T_1}(s, x)}{\partial x^2}, \qquad x \in [0, l_1], \tag{3.1a}$$

$$s\overline{T_j}(s, x) = k_j \frac{\partial^2 \overline{T_j}(s, x)}{\partial x^2}, \qquad x \in [l_1 + \cdots + l_{j-1}, l_1 + \cdots + l_j], \qquad j = 2, \ldots, n, \tag{3.1b}$$

where a bar over function $f(t)$ designates its Laplace transformation on t (e.g. [10]):

$$\bar{f}(s) = \mathcal{L}(f(t)) = \int_0^{\infty} \exp(-s\tau) f(\tau)\, \mathrm{d}t. \tag{3.2a}$$

The following property of Laplace transformation is needed later in our calculation:

$$\bar{g}(s) = \bar{f}_1(s)\bar{f}_2(s) \quad \text{gives} \quad g(t) = \int_0^t f_1(\tau) f_2(t - \tau)\, \mathrm{d}\tau. \tag{3.2b}$$

For the boundary conditions, we obtain:

$$-\lambda_1 \frac{\partial \overline{T_1}}{\partial x}(s, 0) = -\alpha_{\text{out}}(\overline{T_1}(s, 0) - \overline{T_{\text{out}}}(s)), \tag{3.3a}$$

$$\overline{T_j}(s, l_1 + \cdots + l_j) = \overline{T_{j+1}}(s, l_1 + \cdots + l_j), \qquad j = 1, \ldots, n-1, \tag{3.3b}$$

$$-\lambda_j \frac{\partial \overline{T_j}}{\partial x}(s, l_1 + \cdots + l_j) = -\lambda_{j+1} \frac{\partial \overline{T_{j+1}}}{\partial x}(s, l_1 + \cdots + l_j), \qquad j = 1, \ldots, n-1, \tag{3.3c}$$

$$-\lambda_n \frac{\partial \overline{T_n}}{\partial x}(s, l_1 + \cdots + l_n) = -\alpha_{\text{in}}(-\overline{T_n}(s, l_1 + \cdots + l_n)). \tag{3.3d}$$

Solutions $\overline{T_j}$, $j = 1, \ldots, n$ of system (3.1) are found to have the forms (e.g. [1])

$$\overline{T_1} = A_1 \sinh(q_1 x) + B_1 \cosh(q_1 x), \qquad x \in [0, l_1], \tag{3.4a}$$

$$\overline{T_j} = A_j \sinh(q_j(x - l_1 - \cdots - l_{j-1})) + B_j \cosh(q_j(x - l_1 - \cdots - l_{j-1})), \\ x \in [l_1 + \cdots + l_{j-1}, l_1 + \cdots + l_j], \tag{3.4b}$$

where $q_j = \sqrt{\frac{s}{k_j}}$, $j = 1, \ldots, n$. Coefficients A_j, and B_j, $j = 1, \ldots, n$ are determined by the boundary conditions (3.3). Inserting equation (3.4) into equation (3.3) and rearranging the resulting equations and setting $\xi_j = q_j l_j$, $j = 1, \ldots, n$ and $h_j = \frac{\lambda_{j+1}}{\lambda_j}\sqrt{\frac{k_j}{k_{j+1}}}$, $j = 1, \ldots, n-1$ yield the following equations:

$$\lambda_1 q_1 A_1 - \alpha_{\text{out}} B_1 = -\alpha_{\text{out}} \overline{T_{\text{out}}}, \tag{3.5a}$$

$$A_j \sinh \xi_j + B_j \cosh \xi_j - B_{j+1} = 0, \qquad j = 1, \ldots, n-1, \tag{3.5b}$$

$$A_j \cosh \xi_j + B_j \sinh \xi_j - h_j A_{j+1} = 0, \qquad j = 1, \ldots, n-1, \tag{3.5c}$$

$$A_n h_A + B_n h_B = 0, \tag{3.5d}$$

where

$$h_A = \lambda_n q_n \cosh \xi_n + \alpha_{\text{in}} \sinh \xi_n, \qquad h_B = \lambda_n q_n \sinh \xi_n + \alpha_{\text{in}} \cosh \xi_n.$$

The coefficients A_j and B_j, $j = 1, \ldots, n$ can be solved from the linear system (3.5) as follows:

Let $\Delta(s)$

$$= \begin{vmatrix}
\lambda_1 q_1 & -\alpha_{\text{out}} & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\sinh \xi_1 & \cosh \xi_1 & 0 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\cosh \xi_1 & \sinh \xi_1 & -h_1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & \sinh \xi_2 & \cosh \xi_2 & 0 & -1 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & \cosh \xi_2 & \sinh \xi_2 & -h_2 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & \sinh \xi_{n-1} & \cosh \xi_{n-1} & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & \cosh \xi_{n-1} & \sinh \xi_{n-1} & -h_{n-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & h_A & h_B
\end{vmatrix}. \tag{3.6}$$

Coefficients A_j and B_j, $j = 1, 2, \ldots, n$ are obtained by applying Gramer's rule which means that $2j-1$th and $2j$th columns of $\Delta(s)$ are replaced by proper columns, i.e.

$$A_j = \frac{\begin{array}{c} \downarrow \text{ Column } 2j-1 \\ \begin{vmatrix}
\lambda_1 q_1 & -\alpha_{\text{out}} & 0 & -\alpha_{\text{out}} \overline{T_{\text{out}}} & 0 & 0 & 0 & 0 \\
\sinh \xi_1 & \cosh \xi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cosh \xi_1 & \sinh \xi_1 & -h_1 & 0 & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \cosh \xi_{n-1} & \sinh \xi_{n-1} & -h_n & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & h_A & h_B
\end{vmatrix} \end{array}}{\begin{Vmatrix}
\lambda_1 q_1 & -\alpha_{\text{out}} & 0 & \ldots & 0 & 0 & 0 & 0 \\
\sinh \xi_1 & \cosh \xi_1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
s\cosh \xi_1 & \sinh \xi_1 & -h_1 & \ldots & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & \cosh \xi_{n-1} & \sinh \xi_{n-1} & -h_n & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & h_A & h_B
\end{Vmatrix}}$$

$$= \Delta_{2j-1}(s) \overline{T_{\text{out}}}, \tag{3.7a}$$

$$
B_j = \frac{\begin{array}{c} \qquad\qquad\downarrow\ \text{Column } 2j \\ \begin{vmatrix} \lambda_1 q_1 & -\alpha_{\text{out}} & 0 & -\alpha_{\text{out}}\overline{T_{\text{out}}} & 0 & 0 & 0 & 0 \\ \sinh\xi_1 & \cosh\xi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cosh\xi_1 & \sinh\xi_1 & -h_1 & 0 & 0 & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \cosh\xi_{n-1} & \sinh\xi_{n-1} & -h_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & h_A & h_B \end{vmatrix} \end{array}}{\begin{Vmatrix} \lambda_1 q_1 & -\alpha_{\text{out}} & 0 & \ldots & 0 & 0 & 0 & 0 \\ \sinh\xi_1 & \cosh\xi_1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ \cosh\xi_1 & \sinh\xi_1 & -h_1 & \ldots & 0 & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \cosh\xi_{n-1} & \sinh\xi_{n-1} & -h_n & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & h_A & h_B \end{Vmatrix}}
$$

$$
= \Delta_{2j}(s)\overline{T_{\text{out}}}, \tag{3.7b}
$$

where $\Delta_{2j-1}(s)$ and $\Delta_{2j}(s)$, $j = 1, \ldots, n$ are the determinants given above.

Finally, solutions (3.4) can be written as

$$
\overline{T_1} = A_1 \sinh(q_1 x) + B_1 \cosh(q_1 x) = M_1(s, x)\overline{T_{\text{out}}}, \qquad x \in [0, l_1] \tag{3.8a}
$$

$$
\overline{T_j} = A_j \sinh(q_j(x - l_1 - \cdots - l_{j-1})) + B_j \cosh(q_j(x - l_1 - \cdots - l_{j-1})) = M_j(s, x)\overline{T_{\text{out}}}, \\ x \in [l_1 + \cdots + l_{j-1}, l_1 + \cdots + l_j], \qquad j = 2, \ldots, n, \tag{3.8b}
$$

where

$$
M_1(s, x) = \Delta_1(s) \sinh(q_1 x) + \Delta_2(s) \cosh(q_1 x), \tag{3.9a}
$$

$$
M_j(s, x) = \Delta_{2j-1}(s) \sinh(q_j(x - l_1 - \cdots - l_{j-1})) + \Delta_{2j}(s) \cosh(q_j(x - l_1 - \cdots - l_{j-1})), \\ j = 2, \ldots, n. \tag{3.9b}
$$

The above equations (3.8) and (3.9) and the outdoor temperature T_{out} allows us to calculate $\overline{T}_j$, $j = 1, \ldots, n$.

3.2. Solutions of the equations

The superposition technique is used to obtain an analytical solution. This requires that the governing equations be linear. Let us now confine ourselves to T_1. For T_j, $j = 2, \ldots, n$, the same method can be applied. Let us rewrite the outdoor temperature as follows: $T_{\text{out}}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(\omega_k t + \varphi_k)$. This problem of T_1 can be split up into two simpler subproblems such as

(1) outdoor temperature is constant

$$
T_{\text{out}} = a_0 \tag{3.10}
$$

(2) outdoor temperature is

$$
T_{\text{out}} = \sum_{k=1}^{\infty} a_k \cos(\omega_k t + \varphi_k). \tag{3.11a}
$$

For subproblem (1), since the temperature of the boundary is constant and if we ignore the transient term which will eventually die away, the temperature distribution is the solution of the steady-state situation which can be easily obtained from the thermal resistance of the n layers.

For subproblem (2), we need an auxiliary outdoor temperature defined as the conjugate form of T_{out} as T^*_{out}:

$$T^*_{\text{out}} = \sum_{k=1}^{\infty} a_k \sin(\omega_k t + \varphi_k). \tag{3.11b}$$

Let us define

$$\hat{T}_{\text{out}} = T_{\text{out}} + \mathrm{i}T^*_{\text{out}} = \sum_{k=1}^{\infty} a_k \exp[\mathrm{i}(\omega_k t + \varphi_k)]. \tag{3.12}$$

Let $\hat{T}_1(t, x)$ denote the temperature corresponding to outdoor temperature $\hat{T}_{\text{out}}$ and using the linear property of the equations. Clearly

$$T_1(t, x) = \mathrm{Re}(\hat{T}_1(t, x)), \tag{3.13}$$

where Re represents the real part of the function.

Now we concentrate on finding the solution for $\hat{T}_1(t, x)$. Let $m_1(t, x)$ be the inverse Laplace transformation of $M_1(s, x)$, then equation (3.8a) can be rewritten as

$$\overline{\hat{T}_1} = M_1(s, x)\overline{\hat{T}_{\text{out}}} = \bar{m}_1\overline{\hat{T}_{\text{out}}}. \tag{3.14}$$

Using equation (3.2b), we obtain

$$\begin{aligned}\hat{T}_1(t, x) &= \int_0^t m_1(\tau, x)\hat{T}_{\text{out}}(t-\tau)\,\mathrm{d}\tau \\ &= \int_0^\infty m_1(\tau, x)\hat{T}_{\text{out}}(t-\tau)\,\mathrm{d}\tau - \int_t^\infty m_1(\tau, x)\hat{T}_{\text{out}}(t-\tau)\,\mathrm{d}\tau. \end{aligned} \tag{3.15}$$

As m_1 is a bounded function, the second term of equation (3.15) will tend to zero after a long enough time. Therefore, inserting equation (3.12) into equation (3.15) yields

$$\begin{aligned}\hat{T}_1(t, x) &\approx \int_0^\infty m_1(\tau, x)\hat{T}_{\text{out}}(t-\tau)\,\mathrm{d}\tau = \int_0^\infty m_1(\tau, x)\sum_{k=1}^{\infty} a_k \exp[\mathrm{i}(\omega_k(t-\tau) + \varphi_k)]\,\mathrm{d}\tau \\ &= \sum_{k=1}^{\infty} a_k \exp[\mathrm{i}(\omega_k t + \varphi_k)] \underbrace{\int_0^\infty \exp(-\mathrm{i}\omega_k\tau)m_1(\tau, x)\,\mathrm{d}\tau}_{\text{Laplace transform of } m_1 \text{ at } \mathrm{i}\omega_k = M_1(\mathrm{i}\omega_k, x)\text{, see (3.2}a\text{)}} \\ &= \sum_{k=1}^{\infty} a_k \exp[\mathrm{i}(\omega_k t + \varphi_k)]M_1(\mathrm{i}\omega_k, x) \\ &= \sum_{k=1}^{\infty} a_k M_1(\mathrm{i}\omega_k, x)\exp[\mathrm{i}(\omega_k t + \varphi_k)]. \end{aligned} \tag{3.16}$$

Note that there is a mathematical trick in the above calculation. The inverse Laplace transform $m_1(t, x)$ was acting only as a symbolic function. Taking advantage of the mathematical expression of exponential function, $m_1(t, x)$ was replaced by its Laplace transformation which is already available in equation (3.9). This way a complicated residue calculation is avoided.

Combining equations (3.13) and (3.16) gives

$$\begin{aligned} T_1(t, x) &= \mathrm{Re}(\hat{T}_1(t, x)) \\ &= \sum_{k=1}^{\infty} a_k[\mathrm{Re}(M_1(\mathrm{i}\omega_k, x)) \cos(\omega_k t + \varphi_k) - \mathrm{Im}(M_1(\mathrm{i}\omega_k, x)) \sin(\omega_k t + \varphi_k)) \\ &= \sum_{k=1}^{\infty} a_k |M_1(\mathrm{i}\omega_k, x)| \cos(\omega_k t + \varphi_k + \mathrm{d}\varphi_{1,k}), \end{aligned} \tag{3.17a}$$

where

$$\mathrm{d}\varphi_{1,k} = \arctan\left(\frac{\mathrm{Im}(M_1(\mathrm{i}\omega_k, x))}{\mathrm{Re}(M_1(\mathrm{i}\omega_k, x))}\right), \qquad k = 1, \ldots, \infty. \tag{3.17b}$$

Similarly,

$$T_j(t, x) = \sum_{k=1}^{\infty} a_k |M_j(\mathrm{i}\omega_k, x)| \cos(\omega_k t + \varphi_k + \mathrm{d}\varphi_{j,k}) \tag{3.18a}$$

and

$$\mathrm{d}\varphi_{j,k} = \arctan\left(\frac{\mathrm{Im}(M_j(\mathrm{i}\omega_k, x))}{\mathrm{Re}(M_j(\mathrm{i}\omega_k, x))}\right), \qquad k = 1, \ldots, \infty, \quad j = 2, \ldots, n. \tag{3.18b}$$

4. Solutions for more general boundary conditions

In this section we list, without showing all the details, the solution in the case of a more general boundary condition i.e. the case of a periodic indoor temperature change. In this case, the indoor boundary condition reads

$$T_{\mathrm{in}}(t) = b_0 + \sum_{k=1}^{\infty} b_k \cos(\varpi_k t + \phi_k). \tag{4.1}$$

The corresponding equation (3.3d) becomes

$$-\lambda_n \frac{\partial \overline{T_n}}{\partial x}(s, l_1 + \cdots + l_n) = -\alpha_{\mathrm{in}}(\overline{T_{\mathrm{in}}} - \overline{T_n}(s, l_1 + \cdots + l_n)).$$

Therefore the analytical solutions are determined by the sum of T_{out} and T_{in}, which implies (see equations (3.4), (3.5d), (3.7) and (3.8)):

$$\overline{T_1} = M_1(s, x)\overline{T_{\mathrm{out}}} + N_1(s, x)\overline{T_{\mathrm{in}}}, \qquad x \in [0, l_1], \tag{4.2a}$$

$$\overline{T_j} = M_j(s, x)\overline{T_{\mathrm{out}}} + N_j(s, x)\overline{T_{\mathrm{in}}}, \qquad x \in [l_1 + \cdots + l_{j-1}, l_1 + \cdots + l_j], \quad j = 2, \ldots, n. \tag{4.2b}$$

Omitting the details, the analytical solution reads as

$$\begin{aligned} T_1(t, x) = &\sum_{k=1}^{\infty} a_k |M_1(\mathrm{i}\omega_k, x)| \cos(\omega_k t + \varphi_k + \mathrm{d}\varphi_{1,k}) \\ &+ \sum_{k=1}^{\infty} b_k |N_1(\mathrm{i}\varpi_k, x)| \cos(\varpi_k t + \phi_k + \mathrm{d}\phi_{1,k}) \\ &+ \text{steady-state distribution (piecewise line for } a_0 \text{ and } b_0). \end{aligned} \tag{4.3}$$

Temperatures $T_j(t, x)$, $j = 1, \ldots, n$ can be obtained similarly.

By comparing equations (3.17*a*) and (4.3), we observe that the indoor boundary condition leads to additional amplitude and phase changes caused by the indoor excitation. This result is consistent with intuition.

5. Discussion of the solutions

The transient thermal behaviour of an n-layer composite slab is explicitly expressed in equation (4.3). Now we make some observations.

- It is known that any periodic and piecewise continuous function can be approximated by its Fourier expansion, i.e. the function can be expressed as an expansion in terms of an infinite sum of sines and cosines [11]. For example: functions with constant value, square wave, sawtooth wave and semicircle wave can be approximated in terms of the sum of sines and cosines. Therefore, the solution obtained in this paper has a much broader application range than those in the reviewed papers.
- With a periodic excitation of boundary conditions, the temperature variation of any jth layer material is expressed as periodic excitation with attenuated amplitudes and shifted phases (or time lags) which are given by functions $M_j(s, x)$ and $N_j(s, x)$ in equation (4.2). Therefore the functions $M_j(s, x)$ and $N_j(s, x)$, $j = 1, \ldots, n$ are actually acting as 'transfer' functions from which the attenuated amplitudes and time lags can be obtained at any layer.
- As a simple example, let outdoor temperature be a simple sinusoidal functions $T_{\text{out}}(t) = 5.6 - 10.7\cos\left(\frac{2\pi}{365}(t - 20.0)\right)$ °C and indoor temperature be 0 °C. Then the transient temperature distribution of the jth layer material for an n-layer composite slab will be

$$T_j(t, x) = \text{steady-state distribution (determined by the } n\text{-layer slab's thermal resistance)}$$
$$- 10.7\left|M_j\left(\mathrm{i}\frac{2\pi}{365}, x\right)\right| \cos\left(\frac{2\pi}{365}(t - 20.0) + \mathrm{d}\varphi_j\right), \qquad j = 1, \ldots, n. \tag{5.1}$$

It is easily seen that

$$\text{amplitude} = 10.7\left|M_j\left(\mathrm{i}\frac{2\pi}{365}, x\right)\right|, \qquad j = 1, \ldots, n.$$
$$\text{time lag} = \mathrm{d}\varphi_j, \qquad j = 1, \ldots, n.$$

Time-dependent heat flux can also be explicitly expressed by using equation (5.1).

- Observing the technique we have used in obtaining the analytical solution, it is required that t is large enough to guarantee the second term in equation (3.15) tends to zero. This means that the solution is valid a long time after the beginning. However, an assessment of the comparison for analytical and numerical calculations shows that, amazingly enough, the analytical solution agrees with the numerical almost from the beginning.
- From a practical computational point of view, it is easily seen that the developed method is a very useful method. The calculation involves only computations of determinants which can be easily accomplished by commercial software packages, such as Maple, Matlab, Mathematica and even with just a pencil and paper. Moreover, for any jth layer material, only three matrices are involved. Therefore, the developed method is very suitable for engineering designers for evaluating thermal performances of composite materials using perhaps calculators only.

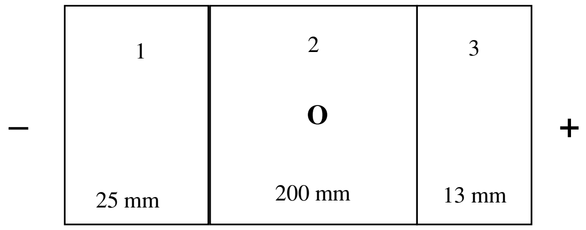

Figure 1. Schematic diagram of the three-layer composite slab studied in example 1.

Table 1. Material properties of the composite slab (see figure 1).

Material	Thermal conductivity (W m^{-1} K^{-1})	Thermal diffusivity (m^2 s^{-1})	Thickness (mm)
1. Wall paper	0.12	1.5×10^{-7}	25
2. Mineral wool	0.0337	1.47×10^{-6}	200
3. Gypsum board	0.23	4.11×10^{-7}	13

Table 2. Average temperature variation in the Helsinki area [12].

Month	Average temperature (°C)	Maximum temperature (°C)	Minimum temperature (°C)
1	−4.2	−1.7	−6.9
2	−4.9	−2.2	−7.7
3	−1.5	1.2	−4.2
4	3.3	6.8	0.4
5	9.9	14.0	6.0
6	14.8	18.7	11.0
7	17.2	20.9	13.7
8	15.8	19.3	12.6
9	10.9	13.9	8.1
10	6.2	8.6	3.8
11	1.4	3.6	−0.8
12	−2.2	0.2	−5.0

6. Examples

This section focuses on the demonstration of the analytical solutions. Two examples are presented here. The specific objective is to assess the accuracy of the analytical method by comparing analytical and numerical solutions.

6.1. Example 1

The selected three-layer slab is the wall structure of our test building. The main material is mineral wool (200 mm). Boundary layers are constructed with wall paper (25 mm) and gypsum board (13 mm). Figure 1 shows a schematic picture of the wall structure and table 1 summarizes the properties of the materials.

Calculation of transient temperature change was made over the central region of the mineral wool (1D and marked as O). The boundary condition, outdoor temperature, was taken from measured monthly weather statistics from 1971 to 2000 in the Helsinki area (see table 2

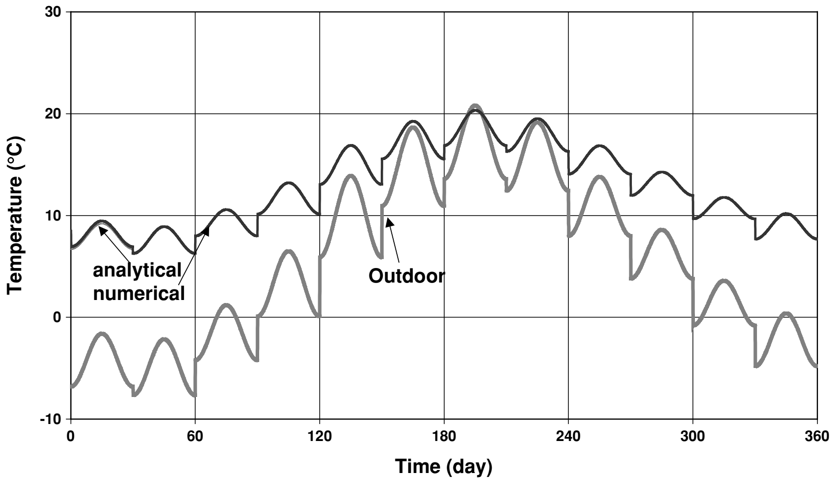

Figure 2. Comparison of analytical and numerical results for the three-layer composite slab shown in figure 1. Data stored in hourly values and shown in daily values.

and [12]). The statistics data were fitted with cosine functions with periods 30 and 365 days. Indoor temperature was assumed to be 20 °C. Throughout all the calculations, convective heat transfer coefficients were assumed to be $\alpha_{\text{out}} = 25$ W m^{-2} K^{-1} and $\alpha_{\text{in}} = 6$ W m^{-2} K^{-1}.

Figure 2 presents the comparison of the transient temperature variation using the analytical and numerical methods. The temperatures were stored in files as hourly values and shown in figures as hourly and daily values. The maximal discrepancy is about 0.2 °C which appears in the first month with relative error of 1%. During the next 11 months, the discrepancies are within 0.008 °C (relative error of 0.04%). As described earlier, the initial value was not specified in developing the analytical solutions. Therefore the initial temperature was not known in the numerical calculation. Here in all the examples, initial temperatures were simply assumed to be linear combination of the average indoor and outdoor temperatures. This resulted in a 1.8 °C discrepancy at time $t = 0$. It is believed that most discrepancies were due to the rough guess of the initial temperature.

The bigger discrepancies of the analytical and numerical results (the result of the first day) are shown in more detail as hourly values in figure 3. The numerical value quickly approaches the analytical value. After 5 h, data show discrepancy of 0.2 °C. Note that the thermal time constant of the three-layer slab is more than 12 h, so the quasi-steady state was certainly not achieved in first 5 h. The accuracy of the analytical method is further demonstrated for another 70 h calculation for the second month in figure 4.

The validation of the numerical programme can be found in [13–15]. In this case, experimental work was also carried out for the verification of the numerical model shown in figure 5.

6.2. *Example 2*

Two more layers are added in the selected structure of example 1. The main constructions are mineral wool, concrete and plywood. Boundary layers are kept the same. Figure 6 demonstrates schematically the new structure and table 3 summarizes the thermal properties of the new added materials.

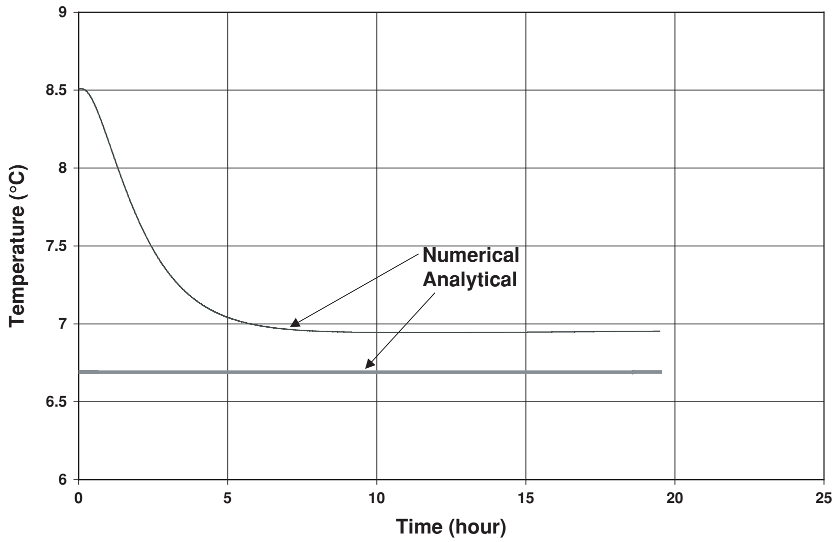

Figure 3. Results of first 20 h from figure 2.

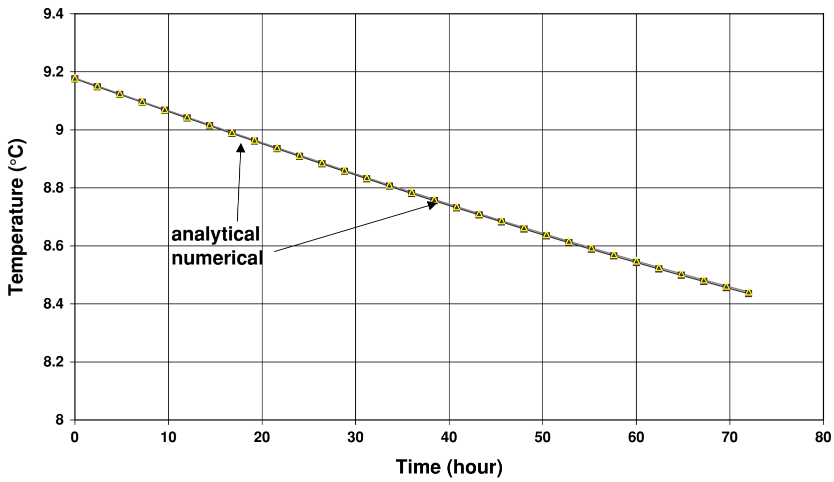

Figure 4. Results of 70 h in the second month from figure 2.

Table 3. Properties of the new materials (see figures 6 and 1).

Material	Thermal conductivity ($W\ m^{-1}\ K^{-1}$)	Thermal diffusivity ($m^2\ s^{-1}$)	Thickness (mm)
4. Concrete	0.9	3.75×10^{-7}	100
5. Plywood	0.147	1.61×10^{-7}	100

In this example, the calculation of transient temperature change was made in the middle region of the mineral wool (1D and marked as O). The boundary temperatures (both outdoor

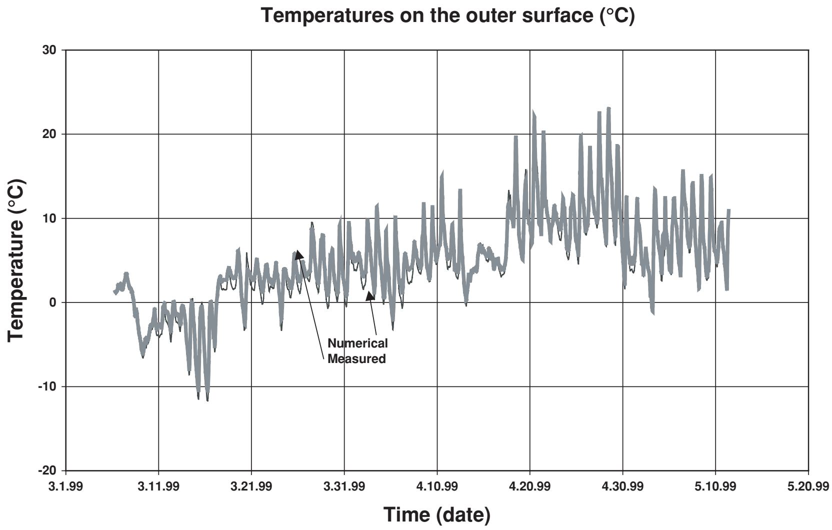

Figure 5. Comparison of numerical results and measurements.

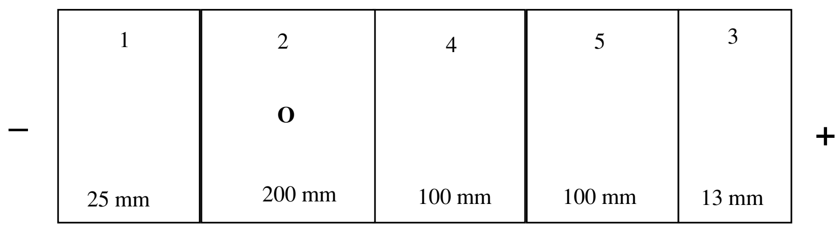

Figure 6. Schematic diagram of the five-layer composite slab studied in example 2. Material properties are listed in tables 1 and 3.

and indoor) were taken from the measurements in figure 5 and then fitted with cosine functions with periods 120, 30, 10, 5 and 1 days with the following type:

$$T_{\text{out}} = a_0 + \sum_1^5 a_i \cos\left(\frac{2\pi t}{\omega_i} - \varphi_i\right), \tag{6.1a}$$

$$T_{\text{in}} = b_0 + \sum_1^5 b_i \cos\left(\frac{2\pi t}{\omega_i} - \phi_i\right), \tag{6.1b}$$

where $\omega_i = 120$, 30, 10, 5 and 1 (day), $i = 1, \ldots, 5$.

The corresponding parameters are listed in table 4 and the fitted temperatures are displayed in figure 7.

Figure 8 shows temperatures in analytical solutions and the temperatures of the numerical models. The results were saved as hourly values in files and shown as daily values in the figure. The discrepancies are within 0.9 °C (relative error 4.8%) during this two-month period. The results of the first 10 h are also shown in figure 9.

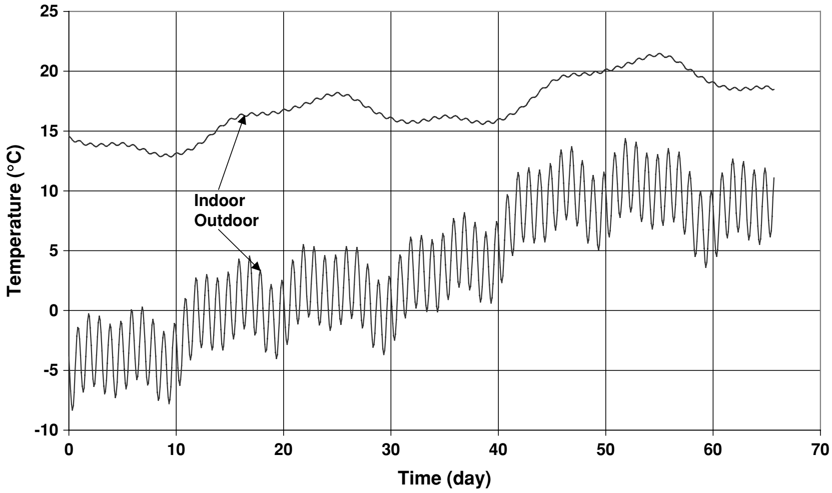

Figure 7. Indoor and outdoor temperature variations.

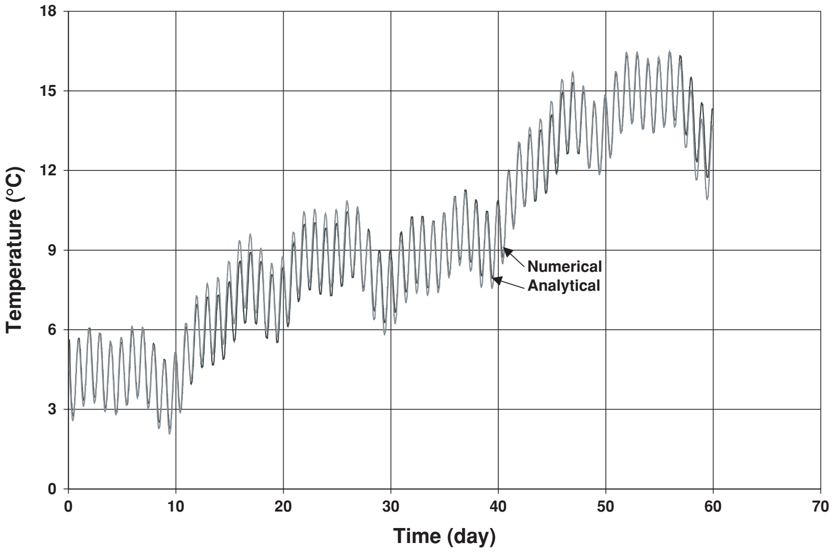

Figure 8. Comparison of analytical and numerical results for the five-layer composite slab shown in figure 6. Data stored in hourly values and shown in daily values.

Table 4. Parameters of equation (6.1).

Parameters	b_0	b_1, ϕ_1	b_2, ϕ_2	b_3, ϕ_3	b_4, ϕ_4	b_5, ϕ_5
Indoor	17.298 92	2.3712, 28.566 93	−1.774 64, 7.790 886	−0.533 07, 6.483 743	−0.053 64, 3.647 276	−0.121 07, 3.295 951
Parameters	a_0	a_1, φ_1	a_2, φ_2	a_3, φ_3	a_4, φ_4	a_5, φ_5
Outdoor	3.607 36	−6.460 41, 31.521 78	1.6039, −8.880 76	−1.083 67, 5.767 917	1.005 466, 2.189 099	−3.068 47, 2.212 049

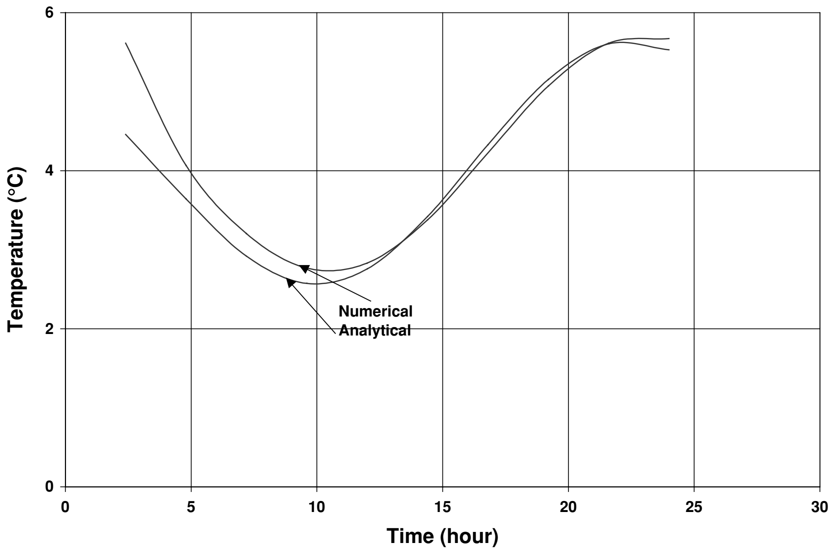

Figure 9. Results of first 10 h from figure 8.

It is worth mentioning that research into other calculation points did not reveal any substantial difference. Therefore we only demonstrated the results for the central region of the mineral wool to save space.

7. Conclusions

In this paper, an analytical approach to heat conduction or mass diffusion in a composite slab subject to general periodic temperature changes has been presented. A hallmark of the result is its simple and concise mathematical forms of the solutions. Agreement with numerical solutions is good. However, in a general conduction or diffusion application context, solutions were only available for constant boundary temperatures for three-layer slab and numerical schemes were usually necessary. The proposed approach is free of these restrictions. The explicit solutions provide insight into the interplay between amplitude decays, time lags and other physical parameters, and can lead to better understanding the thermal process in a composite slab.

The final conclusion to be drawn is that the analytical solutions are really accurate even though, as an approximation formula was used in deriving the solutions in equation (3.16). Comparing analytical and numerical results shows that analytical solutions are accurate almost from the beginning of the calculation time. This suggests that the magnitude of second term in (3.15) is very small. Study of this topic will be left for another paper.

Acknowledgment

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